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Joint Production and the Structure of Technology: A Generalization

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In this paper, we have a peek inside the black box of technology in an attempt to get a better understanding of the concept of joint production. We introduce the notion of input and output subtechnologies; these are then used as building blocks to define various types of production processes, either joint or nonjoint. Thus, in the 2×2 case, we are able to identify up to 36 different production structures, some of which are well known, but most of which are new. These are all described in the primal quantity space as well as in the dual price space. Comparative statics results for the 2×2 joint production process are derived.

Keywords: technology, nonjointness, joint production, activity analysis, network technology.

JEL Classification: F11, D24.

"An Economist is someone who cannot see something working in practice without asking whether it would work in theory."

Anonymous[1]

1 Introduction

Considerable attention has recently been devoted to the concept of fragmentation, i.e., the break-up of production, which makes it possible to

1 From *A Dictionary of Economic Quotations* compiled by Simon James as quoted in the 1988 Heffer catalogue.

take full advantage of international comparative advantages and which reinforces the trend towards the globalization of economic activities.[2] This focus on fragmentation has been accompanied by a regain of interest in the structure of production, and in understanding the ways inputs are transformed into outputs.

This paper examines the structure of technology in a multiple-input multiple-output setting, with special emphasis on the concept of joint production. Joint production is often treated somewhat mysteriously, as the outcome of a process buried inside a black box. Our aim is to describe what could be happening inside this box, to show how inputs can be combined to jointly produce several outputs, and to demonstrate that even joint technologies can be broken up, i.e., fragmented.

To this aim, we introduce the concepts of input and output subtechnologies (or subprocesses), and we hypothesize the existence of an array of factor services, or middle products. This makes it easier to understand how inputs can be transformed into outputs. We consider three types of functional relationships for both the input and the output tiers. These are then used as building blocks which can be assembled in different ways. This exercise allows us to identify various types of production processes, some of which are well known, and many of which are new. In fact, it appears that these building blocks, or modules, are so basic that they can be used to generate a wide variety of production structures, including what is usually meant by joint production.

Our approach also makes it possible to analyze nonjoint production in a systematic way. In the two-input two-output case alone, we are able to define eight types of nonjointness. If we allow for asymmetrical situations (i.e., cases where the two inputs or the two outputs are treated differently), the total number of possible cases rises to 36. We show how a joint technology can degenerate to become nonjoint. Nonjointness results from additional restrictions on the structure of the technology; in turn, it can have remarkable implications, such as the Stolper-Samuelson (1941), and the Rybczynski (1955) theorems. Simulations are used to show how some of these results can emerge as the joint technology approaches the limiting nonjointness cases. All structures identified in this paper can be studied in the primal quantity space, or in the dual price space, and they can be implemented empirically.

2 See Jones (2000), and Arndt and Kierzkowski (2001), for instance. An early formal model of fragmentation and trade is provided by Sanyal and Jones (1982).

Throughout this paper we assume constant returns to scale, competitive behavior, and profit maximization (or cost minimization). We assume that all markets exist, and that all prices are observable. Alternatively, and equivalently, we could assume a centralized decision making process where decisions are made so as to maximize national product given factor endowments and output prices, or minimize national income given output quantities and factor prices. Under these conditions, it is well known that intermediate products can be netted out, and that the technology can be described in terms of primary inputs and net outputs exclusively.[3] This viewpoint is often adopted to justify the exclusion of intermediate goods. Our approach here is precisely the opposite. We may well have a technology without intermediate goods, but we introduce a set of virtual middle products to show how simple input and output subtechnologies can be used as building blocks to generate a large variety of aggregate technologies.

Our treatment is related to the activity analysis literature, and in particular to the network technology approach.[4] This approach models the aggregate technology by decomposing it into a series of subprocesses, which are connected among themselves by distribution and collection nodes. This very flexible setting makes it possible to describe fairly intricate technologies, to include intermediate goods and services, to allow for joint products, and to model externalities. There are at least three important differences between our treatment and the network model, though. For on thing, we limit our attention here to two-tier technologies, whereas the literature on network technologies routinely allows for an arbitrary number of subprocesses. Second, the activity analysis literature often allows for production to take place over several periods, whereas our approach here is purely static. On the other hand, we allow for – indeed we mostly focus on – nonlinear subtechnologies, including neoclassical production and factor requirements functions, whereas the network technology approach relies typically on linear relationships.

The remainder of this paper is organized as follows. Section 2 briefly reviews the description of the aggregate technology by the joint revenue and the joint cost functions, and it defines the concept of joint technology. Input and output subtechnologies are introduced in Sect. 3, and a dual

3 See Woodland (1982), for instance.

4 See Shephard and Färe (1980), Färe and Primont (1995), and Färe and Grosskopf (1996), for example.

treatment is given in Sect. 4. Nonjointness is characterized in terms of the joint revenue and joint cost functions in Sect. 5, and mixed cases of nonjointness are analyzed in Sect. 6. Sections 7–8 are devoted to joint production: Sect. 7 characterizes jointness in terms of input and output subtechnologies, while Sect. 8 derives the comparative statics in the general case of joint production in a two-by-two setting; some numerical simulation results are also reported. Section 9 concludes.

2 Aggregate Description of the Technology

Let us consider an activity that uses J inputs (e.g., primary factors) to produce I outputs (e.g., commodities). We can think of an activity as representing the work of a single firm, of an industry, or even of a nation. There may be separate production lines within this activity, or the production of the I outputs can be intricately linked. These ideas will be clarified later on. We denote the vector of input quantities by $\mathbf{x} \equiv [x_1, \ldots, x_j, \ldots, x_J]'$ and the vector of output quantities by $\mathbf{y} \equiv [y_1, \ldots, y_i, \ldots, y_I]'$. Let T be the production possibilities set, i.e., the set of all feasible input and output combinations $(\mathbf{y}, \mathbf{x})$. We assume that T is a convex cone. Let $\mathbf{w} \equiv [w_1, \ldots, w_j, \ldots, w_J]'$ and $\mathbf{p} \equiv [p_1, \ldots, p_i, \ldots, p_I]'$ be the vectors of input and output prices, respectively. Assuming profit maximization, it is well known that the technology can be described by a *joint revenue function* defined as follows (Shephard, 1970; Lau, 1972; Diewert, 1974; McFadden, 1978):[5]

$$R(\mathbf{p}, \mathbf{x}) \equiv \max_{y}\{\mathbf{p}'\mathbf{y} : (\mathbf{y}, \mathbf{x}) \in T\}. \tag{1}$$

Given the assumptions on T, $R(\cdot)$ is well defined for all positive prices and nonnegative input quantities. It is linearly homogeneous, nondecreasing, and convex in prices, and it is linearly homogeneous, nondecreasing, and concave in input quantities. We also assume that $R(\cdot)$ is twice continuously differentiable.

Alternatively, the technology can be described by the *joint cost function* defined as follows (Shephard, 1970; Hall, 1973; McFadden, 1978):

5 Shephard (1970) also uses the term *benefit function*. Alternative representations of the technology would include the *joint production function* and the *distance function*; see Shephard (1970), and Färe and Grosskopf (1996), for instance.

$$C(\mathbf{y}, \mathbf{w}) \equiv \min_{x}\{\mathbf{w}'\mathbf{x} : (\mathbf{y}, \mathbf{x}) \in T\}. \tag{2}$$

Given the assumptions on T, $C(\cdot)$ is well defined for all positive input prices and nonnegative output quantities. $C(\cdot)$ is linearly homogeneous, nondecreasing and concave in the components of $\mathbf{w}$, and it is linearly homogeneous, nondecreasing and convex in the components of $\mathbf{y}$; it is also assumed to be twice continuously differentiable.[6]

Let $[R]$ and $[C]$ be the Hessians of $R(\cdot)$ and $C(\cdot)$, respectively. They can be written as follows:

$$[R] \equiv \begin{bmatrix} R_{pp} & R_{px} \\ R_{xp} & R_{xx} \end{bmatrix},$$

$$[C] \equiv \begin{bmatrix} C_{yy} & C_{yw} \\ C_{wy} & C_{ww} \end{bmatrix},$$

where R_{pp} is the sub-Hessian of $R(\cdot)$ with respect to the components of $\mathbf{p}$ ($R_{pp} \equiv [\partial^2 R(\cdot)/(\partial p_i \partial p_h)]$), and so on. It immediately follows from the properties of $R(\cdot)$ and $C(\cdot)$ that the sub-Hessians of these two functions must satisfy the following conditions:

Proposition 1 (Conditions for the joint revenue function):

(i) The curvature properties of $R(\cdot)$ imply that R_{pp} is positive semi-definite, and that R_{xx} is negative semi-definite.
(ii) It follows from the homogeneity of $R(\cdot)$ that $R'_{pp}\mathbf{p} = \mathbf{0}_I$, $R'_{xx}\mathbf{x} = \mathbf{0}_J$, $R'_{px}\mathbf{x} = \mathbf{y}$, $R'_{xp}\mathbf{p} = \mathbf{w}$.
(iii) Young's Theorem implies that R_{pp} and R_{xx} are symmetric, and that $R_{xp} = R'_{px}$.

Proposition 2 (Conditions for the joint cost function):

(i) The curvature properties of $C(\cdot)$ imply that C_{yy} is positive semi-definite, and that C_{ww} is negative semi-definite.
(ii) It follows from the homogeneity of $C(\cdot)$ that $C'_{yy}\mathbf{y} = \mathbf{0}_I$, $C'_{ww}\mathbf{w} = \mathbf{0}_J$, $C'_{yw}\mathbf{w} = \mathbf{p}$, $C'_{wy}\mathbf{y} = \mathbf{x}$.

6 The joint cost minimization problem is the dual of the joint revenue maximization problem; see Samuelson (1953–54), Shephard (1970), and Diewert and Woodland (1977).

(iii) Young's Theorem implies that C_{yy} and C_{ww} are symmetric, and that $C_{yw} = C'_{wy}$.

These properties are perfectly general, and they must be satisfied by any well behaved technology. Hall (1973) and Kohli (1983) have shown that in the case of *nonjoint production*, additional restrictions on the Hessians of $R(\cdot)$ and/or $C(\cdot)$ can be found. Proposition 1 (Proposition 2) implies that R_{pp} (C_{yy}) and R_{xx} (C_{ww}) are *at most* of rank $I-1$ and $J-1$, respectively. If R_{pp} (C_{yy}) and R_{xx} (C_{ww}) are indeed of rank $I-1$ and $J-1$, we will say that production is joint.[7] If R_{pp} (C_{yy}) and R_{xx} (C_{ww}) are of lesser rank, we will say that production involves elements of non-jointness or disjointness.[8]

Definition (Joint Technology): The aggregate technology is said to be *joint* if the sub-Hessians of the joint revenue function (joint cost function) R_{pp} (C_{yy}) and R_{xx} (C_{ww}) are of rank $I-1$ and $J-1$, respectively.

3 Input and Output Subtechnologies

So far we have said nothing about the structure of the technology. The formulations that we have adopted are perfectly general, and are consistent with joint as well as with nonjoint production. We have treated the technology like a black box into which inputs enter on one side and from which outputs exit on the opposite end. We now propose to take an X-Ray of this box, to get a picture of the circuitry within that links inputs and outputs.

We assume that each of the J factors produces, or generates, or radiates, I factor services (or intermediary goods, or middle products, or characteristics). These may or may not be the same for each factor, so that the total number of services may be as high as $I \times J$. These services are then used to produce the I outputs, each output being produced with the

7 Some authors, e.g., van den Heuvel (1986), use a more general definition of jointness, by calling joint any technology that is not nonjoint in input quantities; see the next section for more details.

8 Similarly, Samuelson (1966) characterizes nonjointness in terms of a singularity theorem. See Kohli (2001) for a an example of a technology that is disjoint.

help of J types of services, one originating from each factor. The *input subtechnologies* can be described by the following factor requirements functions:

$$x_j \geq \tau^j(z_{1j}, \ldots, z_{Ij}), j = 1, \ldots, J \ , \tag{3}$$

where $z_{ij} \geq 0$ represents the amount of service i produced by factor j. These factor services are then used as inputs in the I *output subtechnologies* which are described by the following production functions:

$$y_i \leq \phi^i(z_{i1}, \ldots, z_{iJ}), i = 1, \ldots, I \ . \tag{4}$$

We assume that $\tau^j(\cdot)$ is nondecreasing, linearly homogeneous and quasiconvex, and that $\phi^i(\cdot)$ is nondecreasing, linearly homogeneous and quasi-concave.

Theorem 1: The production possibilities set defined by (3)–(4) is a convex cone.

Proof: See the Appendix.

It follows from Theorem 1 that, assuming profit maximization or cost minimization, the technology defined by (3)–(4) can be represented by the joint revenue function (1) or the joint cost function (2).[9] All middle products or services are thus netted out; moreover, the solution to the centralized optimization problem (1) or (2) is identical to the decentralized one as long as all the intermediate goods markets clear.[10]

We now must be more specific about the forms of (3) and (4). Regarding the input subtechnology, we consider three possibilities:[11]

$$\begin{aligned} &(\mathrm{A}) \quad x_j \geq \textstyle\sum_i z_{ij} \\ &(\mathrm{B}) \quad x_j \geq \max\{z_{1j}, \ldots, z_{Ij}\} \\ &(\mathrm{C}) \quad x_j \geq t^j(z_{1j}, \ldots, z_{Ij}), \end{aligned}$$

9 Note that the converse is not necessarily true. A production process may involve many stages or tiers, so that several layers of middle products might be needed to describe it.

10 See Woodland (1982), for instance.

11 Note that all three are equivalent if $I = 1$. Also note that subtechnology (A) corresponds to a distribution node to use the terminology of Färe and Grosskopf (1996).

where $t^j(\cdot)$ is increasing, linearly homogeneous, and strictly quasiconvex. The factor requirements function is linear under (A), Leontief under (B), and neoclassical under (C).[12] All three factor requirements functions are nondecreasing, linearly homogeneous, and quasiconvex, and they therefore qualify as special cases of (3).

Under subtechnology (A), the input produces (or generates) services (or middle products) which can be used indifferently by any of the I output subtechnologies. This can be interpreted as a situation where a factor is equally good at producing either service, or where it is perfectly mobile between different sectors. Under (B), the input produces services which can be used by the I output subtechnologies simultaneously. This may correspond to a situation where inputs are public, rather than private, and thus can be used by all sectors at the same time; alternatively, it may be that inputs are perfectly immobile between sectors, i.e., sector specific. Input subtechnology (C), finally, is somewhat similar to (A), except that the transformation hyper-surface is strictly concave, rather than linear. In the literature, this case has sometimes been interpreted as one of imperfect factor mobility.[13] It could also be viewed as a case of mixed inputs, in the sense that they have private as well as public characteristics (e.g., congestible public inputs).[14] More generally, it may describe a situation where factors can be engaged in different activities, either simultaneously or over the course of the time period, and where diversity enhances productivity.

Assume $I = 2$. In the upper part of Fig. 1, we show the production (or supply, or allocation) possibilities frontier of the primary factor for given x_j. The frontier is a straight line with a negative slope under (A), a

12 Note that we assume that under (A) and (B) all I services enter each input subtechnology symmetrically. To be more general, we could assume the following factor requirements functions:

$$x_j \geq \sum_i \frac{z_{ij}}{a_{ij}}, \quad x_j \geq \max\left\{\frac{z_{1j}}{a_{1j}}, \frac{z_{2j}}{a_{2j}}, \ldots, \frac{z_{Ij}}{a_{Ij}}\right\},$$

where $a_{ij} > 0$. However, this would have no significant effect on our results, and we therefore assume $a_{ij} = 1$ for simplicity.

13 See Grossman (1981) and Mussa (1982), for instance.

14 See Weymark (1977) for a model of shared consumption along these lines.

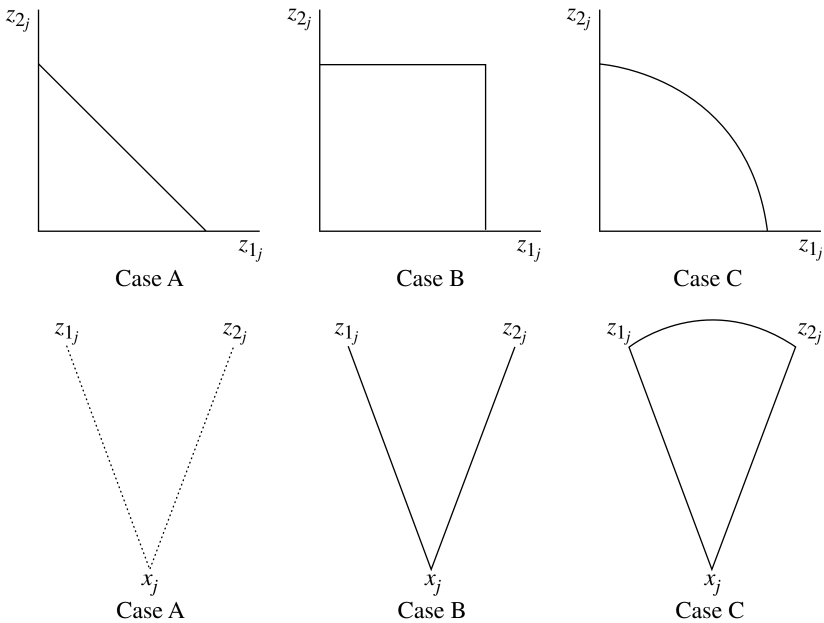

Fig. 1. Input subtechnologies

square box under (B), and strictly concave and downward sloping under (C).

We will also use the diagrams in the lower half of Fig. 1 to describe the three input subtechnologies:[15] a dotted V under (A) to indicate that the factor produces the two services in varying amounts (fixed-sum game), a continuous V under (B) to indicate that the two services are generated simultaneously (no rivalry), and a pie-slice shape under (C) to emphasize that the production possibilities frontier is concave in this case.

We now turn to the output subtechnologies. We again consider three possibilities:[16]

15 These are somewhat similar to the ones used by Jones and Scheinkman (1977).

16 Subtechnology (a) can also be interpreted as a collection node; see Färe and Grosskopf (1996).

$$\begin{array}{ll} \text{(a)} & y_i \leq \sum_j z_{ij} \\ \text{(b)} & y_i \leq \min\{z_{i1}, \ldots, z_{iJ}\} \\ \text{(c)} & y_i \leq f^i(z_{i1}, \ldots, z_{iJ}), \end{array}$$

where $f^i(\cdot)$ is increasing, linearly homogeneous, and strictly quasiconcave. All three production functions are nondecreasing, linearly homogeneous and quasiconcave, and they thus represent special cases of (4). Under (a) the production function is linear, and the output of good i is simply equal to the sum of the corresponding services produced by the J factors. Under (b) the production function is Leontief, and under (c) it is neoclassical.[17]

The three output subtechnologies are represented graphically, for given y_i and assuming $J = 2$, in the top half of Fig. 2. The isoquants are linear under (a), L-shaped under (b), and they have the familiar convex shape under (c). We will also use the simple diagrams depicted in the bottom part of Fig. 2 to identify the three output subtechnologies just described: A dotted inverted V under (a) to indicate that the output originates from either z_{i1} or z_{i2}, a continuous inverted V under (b) to stress that both services are required in fixed proportions to produce the output, and an inverted pie slice under (c) to remind the reader that the isoquant is convex in that case.

Assume for the time being that all J input subtechnologies are of the same type, and similarly for the I output subtechnologies.[18] We can then identify nine different types of production processes:

Aa	**Ab**	**Ac**
Ba	**Bb**	**Bc**
Ca	**Cb**	**Cc**

17 As with the input subtechnologies, we could adopt a more general specification by assuming that, in cases (a) and (b), the J services enter the output subtechnologies in an asymmetrical way:

$$y_i \leq \sum_j \frac{z_{ij}}{b_{ij}}, \qquad y_i \leq \min\left\{\frac{z_{i1}}{b_{i1}}, \ldots, \frac{z_{iJ}}{b_{iJ}}\right\},$$

where $b_{ij} > 0$. We are thus implicitly assuming $b_{ij} = 1$ for simplicity.

18 This assumption, which will be relaxed later on, is made for expositional convenience. Moreover, most models in the literature take this form.

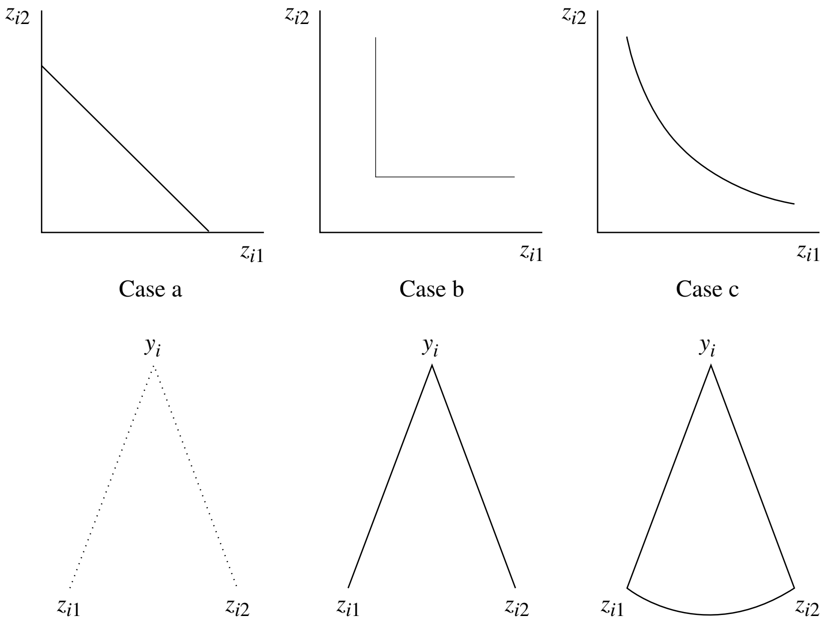

Fig. 2. Output subtechnologies

where the first (bold and capital) letter (**A**, **B**, or **C**) identifies the J input subtechnologies, and the second (bold, lower case) letter (**a**, **b**, or **c**) refers to the I output subtechnologies. These nine production structures can be represented graphically in the 2×2 case using the previously defined symbols. This is done in Fig. 3. We also show in Fig. 4 how the restrictions on the form of the input or output subtechnologies lead to particular forms of nonjointness.

Each one of these nine cases can also be represented mathematically by selecting the appropriate functional forms for the input and output subtechnologies. Thus, case (**Ac**) can be written as:

$$\begin{array}{lll} \text{(A)} & x_j \geq \sum_i z_{ij}, & j = 1, \ldots, J \\ \text{(c)} & y_i \leq f^i(z_{i1}, \ldots, z_{iJ}), & i = 1, \ldots, I. \end{array}$$

Production structure (**Ac**) corresponds to a situation where J private inputs are used in I separate production functions. This is the usual case of nonjointness, more precisely the case of *nonjoint production in input*

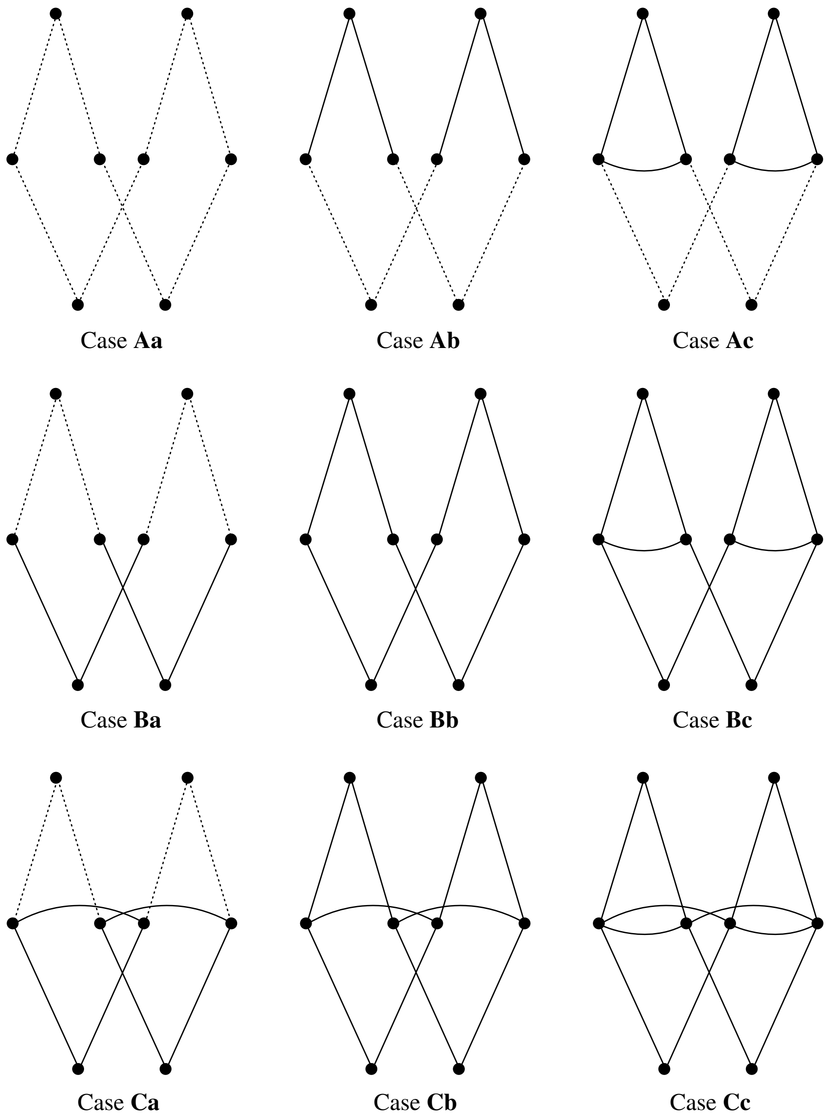

Fig. 3. Production structures in the 2×2 case

quantities to adopt the terminology of Lau (1972), McFadden (1978), Kohli (1981; 1983) and Chambers (1988), and it has been analyzed very extensively in the 2×2 case in the literature on public finance, economic growth, and especially international trade, where it is the backbone of the

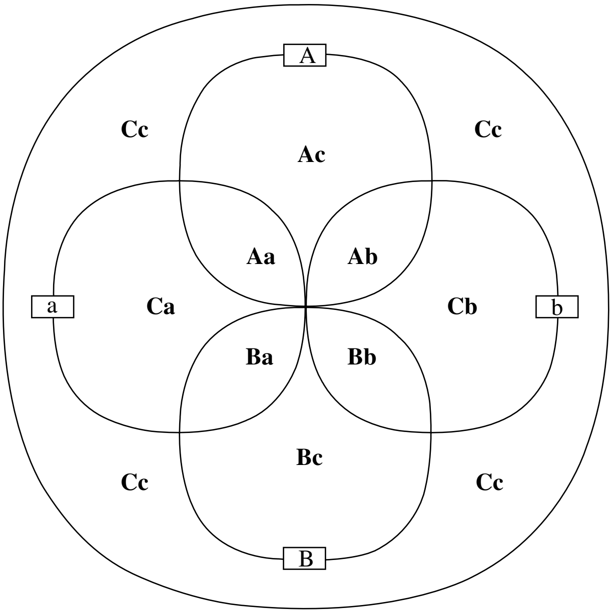

Fig. 4. Input and output subtechnologies, and the structure of production

Heckscher-Ohlin-Samuelson (HOS) model. Structure (**Ab**) too is well known. It is the case of *nonjoint production in input quantities and output prices* (Leontief technology), and it has been discussed by Shephard (1970) and Hall (1973), among others. Structure (**Bc**) is the case of *nonjoint production in input prices* as defined by Kohli (1983; 1985) and Chambers (1988), and it describes a situation where two outputs are produced with two public inputs. Production structure (**Cb**) is the case of *nonjoint production in output prices*, and it has been dubbed the production stations model by Kohli (1994). It describes a situation where each output must be processed by each factor separately, and where each factor is characterized by its own neoclassical transformation function. Structure (**Ca**) corresponds to the case of *nonjoint production in output quantities*, as defined by Lau (1972), Kohli (1983; 1994; 1995), and Chambers (1988), and it fits a situation where each input is fully capable of producing every output on its own. The total supply of outputs is therefore simply equal to the sum of the outputs produced by the individual inputs. The world technology can be viewed as being nonjoint in output quantities if factors of production are immobile internationally, the world supply of goods being equal to the addition of the national supplies.[19]

The remaining production structures are less well known. Production structure (**Ba**) corresponds to the case of *nonjoint production in input prices and output quantities*, and it could be dubbed an anti-Leontief technology. Structures (**Aa**) and (**Bb**) correspond to the cases of *nonjoint production in input and output quantities*, and *nonjoint production in input and output prices*, respectively. The former one has been studied by Ruffin (1988) who views it as a generalization of the Ricardian model. Case (**Cc**), finally, will be discussed in Sects. 7–8 below.

4 The Structure of Production: A Dual Approach

The input and output subtechnologies can also be described in the price space.[20] Let q_{ij} be the – market or shadow – price of service z_{ij}. Assuming that factors maximize their revenues, the input subtechnology can be described by a unit factor revenue function which is dual to the factor requirements function and which is defined as follows:

$$w_j = \varrho^j(\mathbf{q}_{\cdot j}) \equiv \max_{\mathbf{z}_{\cdot j}}\{\mathbf{q}'_{\cdot j}\mathbf{z}_{\cdot j} : 1 \geq \tau^j(\mathbf{z}_{\cdot j})\}, \tag{5}$$

where $\mathbf{z}_{\cdot j} \equiv (z_{1j}, \ldots, z_{Ij})'$ and $\mathbf{q}_{\cdot j} \equiv (q_{1j}, \ldots, q_{Ij})'$. Given the assumptions on $\tau^j(\cdot)$, $\varrho^j(\cdot)$ is nondecreasing, linearly homogeneous and convex in prices (Diewert, 1974).

Assuming cost minimization, the output subtechnology can be described by the following unit cost function which is dual to the output production function:

$$p_i = \chi^i(\mathbf{q}_{i\cdot}) \equiv \min_{\mathbf{z}_{i\cdot}}\{\mathbf{q}'_{i\cdot}\mathbf{z}_{i\cdot} : 1 \leq \phi^i(\mathbf{z}_{i\cdot})\}, \tag{6}$$

where $\mathbf{z}_{i\cdot} \equiv (z_{i1}, \ldots, z_{iJ})'$ and $\mathbf{q}_{i\cdot} \equiv (q_{i1}, \ldots, q_{iJ})'$. Given the assumptions on $\phi^i(\cdot)$, $\chi^i(\cdot)$ is nondecreasing, linearly homogeneous and concave (Diewert, 1974).

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19 Production structure (**Ca**) is also evoked by Jones and Scheinkman (1977) in the 2 × 2 case.

20 See Shephard (1970), Diewert (1974), and McFadden (1978) for a general treatment of the duality between cost and revenue functions.

We now can consider the following three special forms of input subtechnologies:

$$\text{(A)} \quad w_j = \max\{q_{1j}, \ldots, q_{Ij}\}, \quad j = 1, \ldots, J$$

$$\text{(B)} \quad w_j = \Sigma_i q_{ij}, \quad j = 1, \ldots, J$$

$$\text{(C)} \quad w_j = r^j(q_{1j}, \ldots, q_{Ij}), \quad j = 1, \ldots, J,$$

where $r^j(\cdot)$ is increasing, linearly homogeneous, and strictly quasiconvex. Cases (A), (B), (C) are the dual representations of the corresponding cases identified in the previous section. Case (A) indicates that the j-th factor rental price will be equal to the highest service price q_{ij} since the factor can produce any service z_{ij} it wishes ($q_{1j} = q_{2j} = \ldots = q_{Ij}$ if there is diversification). In case (B), the factor rental price is equal to the sum of all corresponding service prices since the factor can produce all z_{ij}'s at once. Case (C), finally, corresponds to the revenue function that is dual to a smooth neoclassical factor requirements function.

On the output side, the three special forms we identified in the previous section can be described in the price space in the following manner:

$$\text{(a)} \quad p_i = \min\{q_{i1}, \ldots, q_{iJ}\}, \quad i = 1, \ldots, I$$

$$\text{(b)} \quad p_i = \Sigma_j q_{ij}, \quad i = 1, \ldots, I$$

$$\text{(c)} \quad p_i = c^i(q_{i1}, \ldots, q_{iJ}), \quad i = 1, \ldots, I,$$

where $c^i(\cdot)$ is increasing, linearly homogeneous, and strictly quasiconcave. The price of output i is equal to the lowest service price q_{ij} in case (a), since any of the J services can be used to produce the output: this is a Leontief cost function, which is dual to a linear production function. The price of i is equal to the sum of all J service prices q_{ij} in case (b), since all J services are needed to produce the output: this is a linear cost function, which is dual to the Leontief production function. In case (c), $c^i(\cdot)$ is the cost function corresponding to a smooth neoclassical production function. In the two-by-two case, the symbols we defined in the previous section could be used to describe the input and output subtechnologies in the price space in the same way as in Figs. 1–3.

5 Characterization of Nonjointness in Terms of the Joint Revenue and Cost Functions

The various cases of nonjointness that we have identified can be characterized in terms of the joint cost function and/or in terms of the joint revenue function.[21] That is, nonjointness implies certain restrictions on the forms of $C(\cdot)$ and/or $R(\cdot)$. These in turn play an important role in shaping comparative statics results. Hall (1973) has shown that a necessary and sufficient condition for nonjointness in input quantities (**Ac**) – the HOS technology – is that the joint cost function can be written as follows:[22]

$$C(\mathbf{w}, \mathbf{y}) = \sum_i y_i C^i(\mathbf{w}), \tag{7}$$

where the $C^i(\cdot)$'s are nondecreasing, linearly homogeneous and concave, whereas Kohli (1983) has shown that a necessary and sufficient condition for nonjointness in output prices (**Cb**) – the production stations technology – is that the joint cost function can be written as:

$$C(\mathbf{w}, \mathbf{y}) = \sum_j w_j h^j(\mathbf{y}), \tag{8}$$

where the $h^j(\cdot)$'s are nondecreasing, linearly homogeneous and convex. Nonjointness in input prices and in output quantities, on the other hand, can best be characterized in terms of the joint revenue function. Kohli (1983) has shown that a necessary and sufficient condition for nonjointness in input prices (**Bc**) – the public inputs technology – is that the joint revenue function can be written as:

$$R(\mathbf{p}, \mathbf{x}) = \sum_i p_i k^i(\mathbf{x}), \tag{9}$$

where the $k^i(\cdot)$'s are nondecreasing, linearly homogeneous and concave, while a necessary and sufficient condition for nonjointness in output

21 See Samuelson (1966) for a characterization of nonjointness in input quantities in terms of the transformation function.

22 This result can be exploited to derive in a straightforward way the Factor-Price Equalization, the Stolper-Samuelson, and the Rybczynski theorems of international trade theory; see Kohli (1991), Sect. 4.4.

quantities (**Ca**) – the world production technology – is that the joint revenue function can be written as follows:

$$R(\mathbf{p},\mathbf{x}) = \sum_j x_j R^j(\mathbf{p}), \tag{10}$$

where the $R^j(\cdot)$'s are nondecreasing, linearly homogeneous and convex.

Shephard (1970) and Hall (1973) have also shown that in the case of nonjointness in input quantities and output prices (**Ab**) – Leontief technology – the joint cost function can be written as:

$$C(\mathbf{w},\mathbf{y}) = \sum_i \sum_j y_i w_j, \tag{11}$$

and it can easily be verified that nonjointness in input prices and output quantities (**Ba**) – anti-Leontief technology – the joint revenue function can be written as:

$$R(\mathbf{p},\mathbf{x}) = \sum_i \sum_j p_i x_j. \tag{12}$$

The case of nonjointness in input and output quantities (**Aa**), on the other hand, implies some rather severe restrictions on both the joint cost and the joint revenue functions. Indeed, it can easily be shown that:

$$C(\mathbf{y},\mathbf{w}) = \min\{w_1,\ldots,w_J\} \sum_i y_i, \tag{13}$$

$$R(\mathbf{p},\mathbf{x}) = \max\{p_1,\ldots,p_I\} \sum_j x_j \ . \tag{14}$$

The case of nonjointness in input and output prices (**Bb**), finally, implies that the joint cost and revenue functions can be written as:

$$C(\mathbf{y},\mathbf{w}) = \max\{y_1,\ldots,y_I\} \sum_j w_j, \tag{15}$$

$$R(\mathbf{p},\mathbf{x}) = \min\{x_1,\ldots,x_J\} \sum_i p_i \ . \tag{16}$$

We can express the restrictions corresponding to the eight forms of nonjointness identified here in terms of the sub-Hessians of the joint cost and/or revenue functions:

$$
\begin{array}{ll}
(\mathbf{Aa}) & C_{yy} = R_{pp} = 0_I, C_{ww} = R_{xx} = 0_J \\
(\mathbf{Ab}) & C_{yy} = 0_I, C_{ww} = 0_J \\
(\mathbf{Ac}) & C_{yy} = 0_I \\
(\mathbf{Ba}) & R_{pp} = 0_I, R_{xx} = 0_J \\
(\mathbf{Bb}) & R_{pp} = C_{yy} = 0_I, R_{xx} = C_{ww} = 0_J \\
(\mathbf{Bc}) & R_{pp} = 0_I \\
(\mathbf{Ca}) & R_{xx} = 0_J \\
(\mathbf{Cb}) & C_{ww} = 0_J,
\end{array}
$$

where 0_I and 0_J are, respectively, $I \times I$ and $J \times J$ null matrices. These restrictions can be imposed and/or tested in empirical work. They also have important implications for the comparative statics results of the model. We summarize these in Tables 1 for the 2×2 case, adopting the revenue function setting, i.e., treating factor endowments and output prices as exogenous as it is the custom in international trade theory.[23] The table shows the impact on output quantities and factor rewards of (i) an exogenous increase in the price of the first output, and (ii) an exogenous increase in the endowment of the first factor.[24] In the case (**Ac**), for example, the table shows that an increase in the price of the first output will not only lead to an increase in the supply of the first output and to reduction in the supply of the other, but it will also lead to an increase in the return of the first factor and to an absolute reduction in the return of the second factor (the Stolper-Samuelson Theorem). Furthermore, an increase in the endowment of the first factor will lead to an increase in the supply of the first output and to an actual reduction in the supply of the other (the Rybczynski Theorem), and it will have no impact on factor rental prices. This last result, which often comes as a surprise to non-trade economists, is a reflection of the Factor-Price Equalization Theorem, and

23 In some instances (i.e., cases (**Ab**), (**Ac**), and (**Cb**)) this requires the inversion of the Hessian of the joint cost function. This is straightforward, however, since at least one of the sub-Hessians is nil in each case. In case (**Ac**), for instance, for $I = J$, $C_{yy} = 0_I$ implies $R_{xx} = 0_J$; see Kohli (1983), Footnote 9. Cases (**Ac**), (**Bc**), (**Ca**), and (**Cb**) are analyzed in greater details in Kohli (1994; 1995).

24 The hats (ˆ) indicate relative changes. For the purpose of this table, we have assumed, for cases (**Ab**) and (**Ac**), that the first output is relatively service-one intensive, and, for case (**Cb**), that the first factor produces service one relatively intensively. For structure (**Ab**), this implies that the b_{ij}'s (see Footnote 17) cannot all be unity: instead $b_{11}/b_{12} > b_{21}/b_{22}$.

Table 1. Nonjointness and comparative statics in the 2×2 production model

Case **Aa**: Nonjointness in input and output quantities

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 = 0,\ \hat{y}_2 = 0; \hat{w}_1 \geq 0, \hat{w}_2 \geq 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 \geq 0, \hat{y}_2 \geq 0; \hat{w}_1 = 0, \hat{w}_2 = 0$$

Case **Ab**: Nonjointness in input quantities and output prices (Leontief model)

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 = 0, \hat{y}_2 = 0; \hat{w}_1 > 0, \hat{w}_2 < 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 < 0; \hat{w}_1 = 0, \hat{w}_2 = 0$$

Case **Ac**: Nonjointness in input quantities (HOS model)

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 < 0; \hat{w}_1 > 0, \hat{w}_2 < 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 < 0; \hat{w}_1 = 0, \hat{w}_2 = 0$$

Case **Ba**: Nonjointness in input prices and output quantities (anti-Leontief model)

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 = 0, \hat{y}_2 = 0; \hat{w}_1 > 0, \hat{w}_2 > 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 > 0; \hat{w}_1 = 0, \hat{w}_2 = 0$$

Case **Bb**: Nonjointness in input and output prices

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 = 0, \hat{y}_2 = 0; \hat{w}_1 \geq 0, \hat{w}_2 \geq 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 \geq 0, \hat{y}_2 \geq 0; \hat{w}_1 = 0, \hat{w}_2 = 0$$

Case **Bc**: Nonjointness in input prices (public inputs model)

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 = 0, \hat{y}_2 = 0; \hat{w}_1 > 0, \hat{w}_2 > 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 > 0; \hat{w}_1 < 0, \hat{w}_2 > 0$$

Case **Ca**: Nonjointness in output quantities (world production model)

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 < 0; \hat{w}_1 > 0, \hat{w}_2 > 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 > 0; \hat{w}_1 = 0, \hat{w}_2 = 0$$

Case **Cb**: Nonjointness in output prices (production stations model)

$$\hat{p}_1 > 0 \rightarrow \hat{y}_1 = 0, \hat{y}_2 = 0; \hat{w}_1 > 0, \hat{w}_2 < 0$$

$$\hat{x}_1 > 0 \rightarrow \hat{y}_1 > 0, \hat{y}_2 < 0; \hat{w}_1 < 0, \hat{w}_2 > 0$$

it is a direct consequence of the assumption of nonjointness in input quantities in the even case.[25]

It is apparent from the above list that input subtechnology (**A**) implies $C_{yy} = 0_I$, and that input subtechnology (**B**) results in $R_{pp} = 0_I$. Similarly, output subtechnology (**a**) implies that $R_{xx} = 0_J$, while output subtechnology (**b**) results in $C_{ww} = 0_J$. Moreover, it is apparent that $C_{yy} = 0_I$ together with $R_{xx} = 0_J$ imply that $C_{ww} = 0_J$ and $R_{pp} = 0_I$, and vice-versa. Finally, it can be seen that input subtechnology (**C**) and output

25 It is ironic that this remarkable feature of the HOS model only holds in the special, even case, whereas it is always true if the production structure is given by (**Aa**),(**Ba**),(**Bb**), or (**Ca**).

subtechnology (**c**) imply no additional restrictions on the sub-Hessians of either $R(\cdot)$ or $C(\cdot)$. Hence, matrices R_{pp} (C_{yy}) and R_{xx} (C_{ww}) are of rank $I-1$ and $J-1$, respectively, and case (**Cc**) therefore designates the case of a joint production technology.

6 Mixed Cases, Public Inputs, and Specific Factors

If we do not restrict our attention to the symmetric cases, that is if we do consider mixed cases, where the input subtechnologies or the output subtechnologies are not the same for the the J inputs or the I outputs, many more cases are possible. Assume that $I = J = 2$; in that case, we can identify 36 different production structures (the numbering of inputs and outputs is irrelevant; hence the number of combinations is six on the input side, and six on the output side). If we now opt for a more detailed notation, where the subtechnologies for each input and output are expressed separately, these possibilities can be expressed as follows:[26]

AAaa	AAab	AAac	AAbb	AAbc	AAcc
ABaa	ABab	ABac	ABbb	ABbc	ABcc
ACaa	ACab	ACac	ACbb	ACbc	ACcc
BBaa	BBab	BBac	BBbb	BBbc	BBcc
BCaa	BCab	BCac	BCbb	BCbc	BCcc
CCaa	CCab	CCac	CCbb	CCbc	CCcc.

Of these 36 cases, 27 are mixed cases. Of those, only a handful have been identified before. Thus, Kohli (1985) defined the case of *almost nonjointness in input prices and quantities* (ABcc) which is described mathematically below and depicted graphically in Fig. 5.

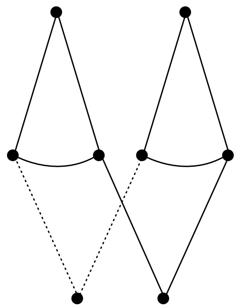
$$\text{(A)} \quad x_1 \geq z_{11} + z_{21},$$

$$\text{(B)} \quad x_2 \geq \max\{z_{12}, z_{22}\},$$

$$\text{(c)} \quad y_i \leq f^i(z_{i1}, z_{i2}), \quad i = 1, 2.$$

This case corresponds to a technology that uses one private input and one public input to produce two outputs with the help of distinct neoclassical production functions. Alternatively, this case can be interpreted as a

26 "Non-mixed" case (AAcc), for instance, is thus equivalent to case (**Ac**), using our earlier "vectorial" notation.



Case Abcc

Fig. 5. Almost nonjointness in input prices and quantities

production process using two specific factors in fixed relative supplies, as well as one mobile factor.[27]

Comparative statics results for this case are reported in Kohli (1985). Kohli (1985) also shows that if we define the following variable (or restricted) profit and cost functions:

$$\pi(\mathbf{p}, w_1, x_2) \equiv \max_{\mathbf{y}, x_1} \{\mathbf{p}'\mathbf{y} - w_1 x_1 : (\mathbf{y}, \mathbf{x}) \in T\}, \tag{17}$$

$$\Gamma(\mathbf{y}, w_1, x_2) \equiv \min_{x_1} \{w_1 x_1 : (\mathbf{y}, \mathbf{x}) \in T\}, \tag{18}$$

then almost nonjointness in input prices and quantities implies the following restrictions:

$$\pi(\mathbf{p}, w_1, x_2) = \sum_i \pi^i(p_i, w_1, x_2), \tag{19}$$

$$\Gamma(\mathbf{y}, w_1, x_2) = \sum_i \Gamma^i(y_i, w_1, x_2). \tag{20}$$

These restrictions can be implemented empirically; see Livernois and Ryan (1989), and Kohli (1993).

Some other mixed cases of nonjointness have been identified before. For instance, Mussa (1982) discusses the case of a two-sector model where one factor is totally immobile between sectors, while the other is less than perfectly mobile; this situation can be described by case (BCcc).

27 See Jones (1971) and Kohli (1993).

Grossman (1983), Casas (1984), and Yu and Parai (1989) analyze a two-sector model where one factor is perfectly mobile, whereas the other is imperfectly mobile; this corresponds to case (ACcc). It is obvious that these and other mixed forms of nonjointness yield restrictions similar to those implied by (19)–(20), but it is beyond the scope of this paper to analyze each case in details.

7 Joint Production

Case **(Cc)** seems particularly interesting because it combines the most complex functional relationships at both the input and the output levels. It is represented graphically at the bottom of Fig. 3 for the 2×2 case, and it has been partially analyzed in that dimension by Hill and Méndez (1983). In the general case, it can be represented mathematically as follows:

$$\text{(C)} \quad x_j \geq t^j(\mathbf{z}_{\cdot j}) \quad j = 1, \ldots, J, \tag{21}$$
$$\text{(c)} \quad y_i \leq f^i(\mathbf{z}_{i\cdot}) \quad i = 1, \ldots, I. \tag{22}$$

Alternatively, using the dual description of the technology given in Sect. 4, we can write:

$$\text{(C)} \quad w_j = r^j(\mathbf{q}_{\cdot j}) \quad j = 1, \ldots, J, \tag{23}$$
$$\text{(c)} \quad p_i = c^i(\mathbf{q}_{i\cdot}) \quad i = 1, \ldots, I. \tag{24}$$

The revenue maximizing supply of services can be obtained by differentiation of the revenue functions:

$$z_{ij} = z^s_{ij}(\mathbf{q}_{\cdot j}, x_j) = \frac{\partial r^j(\cdot)}{\partial q_{ij}} x_j, \tag{25}$$

while the *inverse* supply of services is obtained by differentiating the factor requirements functions:

$$q_{ij} = q^s_{ij}(\mathbf{z}_{\cdot j}, w_j) = \frac{\partial t^j(\cdot)}{\partial z_{ij}} w_j. \tag{26}$$

Similarly, the cost minimizing demand for services can be obtained by differentiation of the cost functions:[28]

28 This result is known as Shephard's Lemma; see Shephard (1953; 1970).

$$z_{ij} = z_{ij}^d(\mathbf{q}_{i\cdot}, y_i) = \frac{\partial c^i(\cdot)}{\partial q_{ij}} y_i, \tag{27}$$

and the *inverse* demand for services is obtained by differentiation of the production functions:

$$q_{ij} = q_{ij}^d(\mathbf{z}_{i\cdot}, p_i) = \frac{\partial f^i(\cdot)}{\partial z_{ij}} p_i. \tag{28}$$

Assume that output prices and input quantities are given. This is the formulation that corresponds to the description of the technology by the joint revenue function. Equilibrium of the model can then be described by (22), (23), (25) and (28). This gives a total of $I + J + 2(I \times J)$ equations to determine I output supplies, J factor prices, and the prices and quantities of the $I \times J$ services. Alternatively, assume that output quantities and input prices are given. Equilibrium can then be described by (21), (24), (26) and (27). This again yields $I + J + 2(I \times J)$ equations to determine I output prices, J factor quantities, and the prices and quantities of the $I \times J$ services.

8 Joint Production: Comparative Statics Results in the 2×2 Case

In this section, we allow for two inputs and two outputs. The two outputs are labeled 1 and 2, but to make the analysis more palatable, we will assume that the two inputs are labor (L) and capital (K). We will consider factor endowments and output prices as given. It is then appropriate to describe the technology by a revenue function such as (1):

$$R(p_1, p_2, x_L, x_K) \equiv \max_{\mathbf{y}}\{p_1 y_1 + p_2 y_2 : (y_1, y_2, x_L, x_K) \in T\}.$$

We next define Θ as the aggregate (i.e., economy-wide) elasticity of transformation between outputs 1 and 2, and Ψ as the aggregate elasticity of complementarity between labor and capital:

$$\Theta \equiv \frac{RR_{12}}{R_1 R_2} \leq 0,$$
$$\Psi \equiv \frac{RR_{LK}}{R_L R_K} \geq 0\ ,$$

where $R_i \equiv \partial R(\cdot)/\partial p_i \quad (i = 1, 2)$, $R_j \equiv \partial R(\cdot)/\partial x_j \quad (j = L, K)$, $R_{12} \equiv \partial^2 R(\cdot)/(\partial p_1 \partial p_2)$, $R_{LK} \equiv \partial^2 R(\cdot)/(\partial x_L\, \partial x_K)$, and $R \equiv R(\cdot)$ for short. Furthermore, we define Δ as a measure of the marginal factor intensities of the two outputs:

$$\Delta \equiv \frac{R^2}{R_1 R_2 R_L R_K}(R_{1L}R_{2K} - R_{1K}R_{2L}),$$

where $R_{ij} \equiv \partial^2 R(\cdot)/(\partial p_i \partial x_j)(i = 1, 2; j = L, K)$. A positive value of Δ indicates that output 1 is relatively labor intensive at the margin.

Finally, let Λ and Ω be the national product shares of output 1 and labor, respectively:

$$\Lambda \equiv \frac{p_1 y_1}{R},$$
$$\Omega \equiv \frac{w_L x_L}{R}.$$

Taking account of the homogeneity and symmetry properties of $R(\cdot)$, the comparative statics of the model can be entirely described in terms of the five parameters $\Theta, \Psi, \Delta, \Lambda$ and Ω:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{w}_L \\ \hat{w}_K \end{bmatrix} = \begin{bmatrix} -\Theta(1-\Lambda) & \Theta(1-\Lambda) & \Omega+\Omega(1-\Omega)(1-\Lambda)\Delta & 1-\Omega-\Omega(1-\Omega)(1-\Lambda)\Delta \\ \Theta\Lambda & -\Theta\Lambda & \Omega-\Omega(1-\Omega)\Lambda\Delta & 1-\Omega+\Omega(1-\Omega)\Lambda\Delta \\ \Lambda+\Lambda(1-\Lambda)\Omega\Delta & 1-\Lambda-\Lambda(1-\Lambda)(1-\Omega)\Delta & -\Psi(1-\Omega) & \Psi(1-\Omega) \\ \Lambda-\Lambda(1-\Lambda)\Omega\Delta & 1-\Lambda+\Lambda(1-\Lambda)\Omega\Delta & \Psi\Omega & -\Psi\Omega \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{x}_L \\ \hat{x}_K \end{bmatrix}, \tag{29}$$

where the hats (ˆ) indicate relative changes.

Using next (23) and (25) as a starting point, the comparative statics of the input subtechnologies can be summarized as follows:

$$\begin{bmatrix} \hat{w}_L \\ \hat{w}_K \\ \hat{z}_{1L} \\ \hat{z}_{1K} \\ \hat{z}_{2L} \\ \hat{z}_{2K} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_L & 0 & 1-\lambda_L & 0 \\ 0 & 0 & 0 & \lambda_K & 0 & 1-\lambda_K \\ 1 & 0 & -\theta_L(1-\lambda_L) & 0 & \theta_L(1-\lambda_L) & 0 \\ 0 & 1 & 0 & \theta_K(1-\lambda_K) & 0 & \theta_K(1-\lambda_K) \\ 1 & 0 & \theta_L\lambda_L & 0 & -\theta_L\lambda_L & 0 \\ 0 & 1 & 0 & \theta_K\lambda_K & 0 & -\theta_K\lambda_K \end{bmatrix} \begin{bmatrix} \hat{x}_L \\ \hat{x}_K \\ \hat{q}_{1L} \\ \hat{q}_{1K} \\ \hat{q}_{2L} \\ \hat{q}_{2K} \end{bmatrix}, \tag{30}$$

where θ_j is the elasticity of transformation of input subtechnology j $(\theta_j \equiv \varrho^j \varrho^j_{12}/(\varrho^j_1 \varrho^j_2) \leq 0)$ and $\lambda_j \equiv (q_{1j}z_{1j})/(q_{1j}z_{1j} + q_{2j}z_{2j}), (j = L, K)$.

It is convenient to partition (30) and to rewrite it in the following simplified form:

$$\begin{bmatrix} \hat{\mathbf{w}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{q}} \end{bmatrix}, \tag{31}$$

where $\hat{\mathbf{w}} \equiv (\hat{w}_L, \hat{w}_K)'$, $\hat{\mathbf{z}} \equiv (\hat{z}_{1L}, \hat{z}_{1K}, \hat{z}_{2L}, \hat{z}_{2K})'$, $\hat{\mathbf{x}} \equiv (\hat{x}_L, \hat{x}_K)'$, $\hat{\mathbf{q}} \equiv (\hat{q}_{1L}, \hat{q}_{1K}, \hat{q}_{2L}, \hat{q}_{2K})'$, and A_{11}, A_{12}, A_{21}, and A_{22} are defined accordingly.

Next, from (22) and (28) we can write the comparative statics for the output side of the model:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{q}_{1L} \\ \hat{q}_{1K} \\ \hat{q}_{2L} \\ \hat{q}_{2K} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_1 & 1-\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_2 & 1-\omega_2 \\ 1 & 0 & -\psi_1(1-\omega_1) & \psi_1(1-\omega_1) & 0 & 0 \\ 1 & 0 & \psi_1\omega_1 & -\psi_1\omega_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\psi_2(1-\omega_2) & \psi_2(1-\omega_2) \\ 0 & 1 & 0 & 0 & \psi_2\omega_2 & -\psi_2\omega_2 \end{bmatrix} \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{z}_{1L} \\ \hat{z}_{1K} \\ \hat{z}_{2L} \\ \hat{z}_{2K} \end{bmatrix}, \tag{32}$$

where ψ_i is the Hicksian elasticity of complementarity of output subtechnology i ($\psi_i \equiv \phi^i \phi^i_{LK}/(\phi^i_L \phi^i_K) \geq 0$) and $\omega_i \equiv (q_{i1}z_{i1}) \ /(q_{i1}z_{i1} + q_{i2}z_{i2})$, $(i = 1, 2)$. Note that it follows from the definitions of the λ's and ω's that they must satisfy the following accounting restriction:

$$\omega_1(1 - \omega_2)(1 - \lambda_L)\lambda_K - (1 - \omega_1)\omega_2\lambda_L(1 - \lambda_K) = 0. \tag{33}$$

Naturally, (32) can be rewritten as:

$$\begin{bmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{z}} \end{bmatrix}, \tag{34}$$

where $\hat{\mathbf{y}} \equiv (\hat{y}_1, \hat{y}_2)'$, $\hat{\mathbf{p}} \equiv (\hat{p}_1, \hat{p}_2)'$, and B_{11}, B_{12}, B_{21}, and B_{22} are the blocks of the square matrix on the right-hand side of (32).

It is then a simple matter to use (31) and (34) to eliminate $\hat{\mathbf{q}}$ and $\hat{\mathbf{z}}$ to solve the system for $\hat{\mathbf{y}}$ and $\hat{\mathbf{w}}$ as functions of $\hat{\mathbf{p}}$ and $\hat{\mathbf{x}}$:

$$\begin{bmatrix} \hat{\mathbf{y}} \\ \hat{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} B_{12}[I - A_{22}B_{22}]^{-1}A_{22}B_{21} & B_{12}[I - A_{22}B_{22}]^{-1}A_{21} \\ A_{12}B_{21} + A_{12}B_{22}[I - A_{22}B_{22}]^{-1}A_{22}B_{21} & A_{12}B_{22}[I - A_{22}B_{22}]^{-1}A_{21} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{x}} \end{bmatrix}, \tag{35}$$

where I is a 4×4 identity matrix. Note that (35) is directly comparable to (29): it thus is possible to use (35) to express the five aggregate parameters ($\Theta, \Psi, \Delta, \Lambda$ and Ω) in terms of the disaggregate parameters of the input and output subtechnologies ($\lambda_L, \lambda_K, \theta_L, \theta_K, \omega_1, \omega_2, \psi_1$ and ψ_2). This yields the following:[29]

$$\begin{aligned}
\Theta &= \frac{\mathscr{A}}{\mathscr{D}}, \\
\Psi &= \frac{\mathscr{B}}{\mathscr{D}}, \\
\Lambda &= \frac{\lambda_L \lambda_K}{\lambda_L(1-\omega_1) + \lambda_K \omega_1}, \\
\Omega &= \frac{\lambda_K \omega_1}{\lambda_L(1-\omega_1) + \lambda_K \omega_1}, \\
\Delta &= \frac{\mathscr{C}/\mathscr{D}}{\Omega(1-\Omega)},
\end{aligned}$$

where:

$$\begin{aligned}
\mathscr{A} = {} & \theta_L[(1-\lambda_L)\omega_1 + \lambda_L\omega_2] + \theta_K[\lambda_K(1-\omega_2) + (1-\lambda_K)(1-\omega_1)] \\
& - \theta_L\theta_K\{\psi_1[(1-\lambda_L)(1-\omega_2) + (1-\lambda_K)\omega_2] \\
& + \psi_2[\lambda_L(1-\omega_1) + \lambda_K\omega_1]\} \leq 0, \\
\mathscr{B} = {} & \psi_1[\lambda_L(1-\omega_1) + \lambda_K\omega_1] + \psi_2[(1-\lambda_L)(1-\omega_2) + (1-\lambda_K)\omega_2] \\
& - \psi_1\psi_2\{\theta_L[\lambda_K(1-\omega_2) + (1-\lambda_K)(1-\omega_1)] \\
& + \theta_K[\lambda_L\omega_2 + (1-\lambda_L)\omega_1]\} \geq 0, \\
\mathscr{C} = {} & (1-\omega_1)\omega_2(\theta_L\psi_1 + \theta_K\psi_2) - \omega_1(1-\omega_2)(\theta_L\psi_2 + \theta_K\psi_1) \\
& + (\omega_1 - \omega_2)(1 + \theta_L\theta_K\psi_1\psi_2) \gtrless 0, \\
\mathscr{D} = {} & 1 - \theta_L\psi_1(1-\lambda_L)(1-\omega_1) - \theta_L\psi_2\lambda_L(1-\omega_2) - \theta_K\psi_1\omega_1(1-\lambda_K) \\
& - \theta_K\psi_2\lambda_K\omega_2 + \theta_L\theta_K\psi_1\psi_2(\lambda_L - \lambda_K)(\omega_1 - \omega_2) \geq 0.
\end{aligned}$$

These expressions can be used to compute the values of Θ, Ψ and Δ for particular forms of nonjointness. For instance, nonjointness in input quantities (the HOS structure of production) obtains when θ_L and θ_K tend to minus infinity. One can thus easily verify that:

29 The algebra tends to become quite tedious; the computations can be facilitated with the help of a computer software program such as MAPLE (Char et al., 1991). Note that use of (33) has been made when deriving the values of Λ and Ω.

$$\lim_{\theta_j \to -\infty} \Theta = -\frac{\psi_1[(1-\lambda_L)(1-\omega_2)+(1-\lambda_K)\omega_2]+\psi_2[\lambda_L(1-\omega_1)+\lambda_K\omega_1]}{\psi_1\psi_2(\lambda_L-\lambda_K)(\omega_1-\omega_2)},$$

$$\lim_{\theta_j \to -\infty} \Psi = 0,$$

$$\lim_{\theta_j \to -\infty} \Delta = \frac{1}{(\lambda_L-\lambda_K)\Omega(1-\Omega)} = \frac{[\lambda_L(1-\omega_1)+\lambda_K\omega_1]}{\lambda_L\lambda_K\omega_1(\lambda_L-\lambda_K)(1-\omega_1)}.$$

Thus, the aggregate elasticity of complementarity (Ψ) is nil. This verifies a well-known property of the HOS model, namely that, for given output prices, an increase in the endowment of either factor has no impact on factor rental prices. This is a result that is particular to the case of nonjointness in input quantities,[30] and it does not hold under joint production. Other forms of nonjointness also lead to some simplifications. Thus the reader can verify that:

$$\lim_{\psi_i \to \infty} \Theta = 0 \qquad \text{(nonjointness in output prices)},$$

$$\lim_{\theta_j \to 0} \Theta = 0 \qquad \text{(nonjointness in input prices)},$$

$$\lim_{\psi_i \to 0} \Psi = 0 \qquad \text{(nonjointness in output quantities)}.$$

Some simulations might be useful to better understand the relationship that exists between the input and output subtechnologies, on the one hand, and the aggregate technology on the other. Thus we report in Table 2 estimates for the price and quantity elasticities of output supply and inverse input demand, for given λ_j's and ω_i's, and for alternative values of the θ_j's and ψ_i's.[31] These elasticities are compatible with the joint revenue function setting; see (29) above. Alternatively, elasticities consistent with the joint cost function setting could be obtained by inverting system (29). Of particular interest are the so-called Rybczynski and Stolper-Samuelson elasticities. It is apparent that as θ_L and θ_K become larger in absolute value, the familiar sign pattern emerges (see the second column). However, the same sign pattern also appears as ψ_1 and ψ_2 take on larger and larger values (column 3); that is, results equivalent to the Rybczynski and Stolper-Samuelson theorems hold under nonjointness in

30 As shown in Sect. 5, nonjointness in input quantities implies $C_{yy} = 0_I$. The fact that it also implies $R_{xx} = 0_J$ is particular to the even case; see footnote 23.

31 The figures for the λ_j's and ω_i's are based on 1987 estimates for the United States, with output disaggregated between investment and consumption goods; see Kohli (1991), p. 250.

Table 2. Joint production in the 2×2 case — Simulation results

	$\lambda_L = 0.300$; $\lambda_K = 0.117$; $\omega_1 = 0.777$; $\omega_2 = 0.519$ ($\Lambda = 0.222$; $\Omega = 0.576$)				
	$\theta_L = -1$ $\theta_K = -1$ $\psi_1 = 1$ $\psi_2 = 1$	$\theta_L = -20$ $\theta_K = -10$ $\psi_1 = 1$ $\psi_2 = 1$	$\theta_L = -1$ $\theta_K = -1$ $\psi_1 = 10$ $\psi_2 = 20$	$\theta_L = -.10$ $\theta_K = -.05$ $\psi_1 = 1$ $\psi_2 = 1$	$\theta_L = -1$ $\theta_K = -1$ $\psi_1 = .05$ $\psi_2 = .10$
Θ	−1.069	−9.948	−0.607	−0.083	−0.963
Ψ	1.069	0.625	10.474	0.963	0.088
Δ	2.374	11.742	12.307	1.162	1.167
$\partial \ln y_1/\partial \ln p_1$	0.831	7.735	0.472	0.065	0.749
$\partial \ln y_1/\partial \ln p_2$	−0.831	−7.735	−0.472	−0.065	−0.749
$\partial \ln y_2/\partial \ln p_1$	−0.238	−2.213	−0.135	−0.018	−0.214
$\partial \ln y_2/\partial \ln p_2$	0.238	2.213	0.135	0.018	0.214
$\partial \ln w_L/\partial \ln x_L$	−0.453	−0.265	−4.440	−0.408	−0.037
$\partial \ln w_L/\partial \ln x_K$	0.453	0.265	4.440	0.408	0.037
$\partial \ln w_K/\partial \ln x_L$	0.616	0.360	6.034	0.555	0.051
$\partial \ln w_K/\partial \ln x_K$	−0.616	−0.360	−6.034	−0.555	−0.051
$\partial \ln y_1/\partial \ln x_L$	1.027	2.806	2.913	0.797	0.798
$\partial \ln y_1/\partial \ln x_K$	−0.027	−1.806	−1.913	0.203	0.202
$\partial \ln y_2/\partial \ln x_L$	0.447	−0.062	−0.092	0.513	0.513
$\partial \ln y_2/\partial \ln x_K$	0.553	1.062	1.092	0.487	0.487
$\partial \ln w_L/\partial \ln p_1$	0.397	1.083	1.125	0.308	0.308
$\partial \ln w_L/\partial \ln p_2$	0.603	−0.083	−0.125	0.692	0.692
$\partial \ln w_K/\partial \ln p_1$	−0.014	−0.947	−1.004	0.107	0.106
$\partial \ln w_K/\partial \ln p_2$	1.014	1.947	2.004	0.893	0.894

output prices; see Kohli (1994). One also can observe how the price elasticities of output supply tend to zero as one moves closer to the cases of nonjointness in input prices (column 4) and nonjointness in output prices (column 3), and how the quantity elasticities of inverse input demand tend to zero as one nears the cases of nonjointness in input quantities (column 2) and nonjointness in output quantities (column 5). The price elasticities of inverse input demand shown at the bottom of column 2 illustrate Samuelson's argument that "... when technology admits of the slightest amount of joint production, the factor-price equalization deduction will be spoiled even for regions that hardly differ in factor endowments ..." (Samuelson, 1992, p. 1).[32]

9 Conclusions

In this paper, we have attempted to have a glimpse inside the black box of technology in order to gain a better understanding of the process of joint production. We have tried to show how various cases of nonjointness, and corresponding restrictions on the joint cost and/or revenue functions, can be obtained by making simplifying assumptions about the shape of the input and output subtechnologies. These restrictions can be tested and/or imposed in empirical work. The various forms of nonjointness may yield some rather remarkable comparative statics results. However, if these results are not matched by reality, they end up being a liability, rather than an asset. Moreover, nonjoint production often leads to difficulties as the number of inputs and outputs increases, particularly in the uneven case. This is not so with joint production where the number of inputs and outputs is irrelevant. What is lost in elegance is more than made up in flexibility and generality. We therefore feel that it is often not worth it to assume nonjoint production. Nevertheless, the study of the various forms of nonjointness has proved useful in allowing us to get a better grip of the joint production process.

The production structures defined in this paper can, in principle, be implemented empirically. Thus, multiple-input multiple-output technologies allowing for joint production have been estimated extensively in

32 As shown by Woodland (1977) and Jones (1992), however, the Factor-Price Equalization Theorem may nevertheless hold under joint production if the number of shared production *activities* is as large as the number of nontraded productive inputs.

recent years, often using joint cost functions and variable profit functions; see Kohli (1991) for a number of examples using U.S. data, including a number of 2×2 models. Estimation of the aggregate technology subject to various forms nonjointness is less frequent. An early example is provided by Burgess (1976); other examples include Kohli (1981; 1991; 1993; 2001), and Livernois and Ryan (1989).

It is also noteworthy that the production structures depicted in this paper could be useful in applied general equilibrium analysis, where researchers often rely on CES and CET functional forms, and thus they often work with single-output production functions or single-input factor requirements functions. Yet the assumption of nonjoint production may be unrealistic in many cases. By allowing for the presence of input and output subtechnologies, and thus by introducing middle products, it becomes possible to model joint production, and yet to continue using the CES/CET functions as building blocks.

We should stress that the picture of the joint production process that we have given in this paper is probably not the only one that one could imagine, maybe not even so in the two-by-two case. What we have attempted to do is to give one possible interpretation of joint production, one that is particularly simple and tractable, but if one allows for many factors, many intermediate goods and services, and many outputs, almost anything could happen inside the black box of technology.

Although middle products were introduced here solely for the purpose of describing how primary inputs were combined into outputs, the framework that we have proposed could be used to analyze the role of intermediate goods in a simple production model. Many additional issues of interest then arise, including questions of trade in middle products, taxation, and welfare. Some of these may be tackled in future research.

Appendix

Proof of Theorem 1

Let T' be the production possibilities set defined by (3)–(4).
Let $\mathbf{y}^A \equiv (y_1^A, \ldots, y_I^A)'$ and $\mathbf{x}^A \equiv (x_1^A, \ldots, x_J^A)'$ be such that $(\mathbf{y}^A, \mathbf{x}^A) \in T'$. By assumption, there exists $(z_{11}^A, \ldots, z_{ij}^A, \ldots, z_{IJ}^A)$ such that $x_j^A \geq \tau^j(z_{1j}^A, \ldots, z_{Ij}^A)$ and $y_i^A \leq \phi^i(z_{i1}^A, \ldots, z_{iJ}^A)$. Linear homogeneity of

$\tau^j(\cdot)$ and $\phi^i(\cdot)$ implies that $\lambda x_j^A \geq \tau^j(\lambda z_{1j}^A, \ldots, \lambda z_{Ij}^A)$ and $\lambda y_i^A \leq \phi^i(\lambda z_{i1}^A, \ldots, \lambda z_{iJ}^A)(\lambda > 0)$. Hence $(\lambda \mathbf{y}^A, \lambda \mathbf{x}^A) \in T'$ which proves that T' is a cone.

Let $\mathbf{y}^A \equiv (y_1^A, \ldots, y_I^A)', \mathbf{y}^B \equiv (y_1^B, \ldots, y_I^B)', \mathbf{x}^A \equiv (x_1^A, \ldots, x_J^A)'$ and $\mathbf{x}^B \equiv (x_1^B, \ldots, x_J^B)'$ be such that $(\mathbf{y}^A, \mathbf{x}^A) \in T'$ and $(\mathbf{y}^B, \mathbf{x}^B) \in T'$. By assumption, there exists $(z_{11}^A, \ldots, z_{ij}^A, \ldots, z_{IJ}^A)$ and $(z_{11}^B, \ldots, z_{ij}^B, \ldots, z_{IJ}^B)$ such that $x_j^A \geq \tau^j(z_{1j}^A, \ldots, z_{Ij}^A), x_j^B \geq \tau^j(z_{1j}^B, \ldots, z_{Ij}^B), y_i^A \leq \phi^i(z_{i1}^A, \ldots, z_{iJ}^A)$ and $y_i^B \leq \phi^i(z_{i1}^B, \ldots, z_{iJ}^B)$. Let $\mathbf{y}^C \equiv \delta \mathbf{y}^A + (1-\delta)\mathbf{y}^B, \mathbf{x}^C \equiv \delta \mathbf{x}^A + (1-\delta)\mathbf{x}^B$ $(0 < \delta < 1)$. Convexity of $\tau^j(\cdot)$ implies that $x_j^C \geq \tau^j(z_{1j}^C, \ldots, z_{Ij}^C)$ and concavity of $\phi^i(\cdot)$ implies that $y_i^C \leq \phi^i(z_{i1}^C, \ldots, z_{iJ}^C)$. Hence $(\mathbf{y}^C, \mathbf{x}^C) \in T'$ which proves that T' is convex.

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